ALM - Basic Interest Rate Theory

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Course Program

- `Basic interest rate theory`
- Interest rate risk management
- Stochastic term structure models
- Risk measurement
- Reinsurance and insurance-linked securities
- Mean-variance analysis for ALM

Contents of the chapter

- A continuous model for yield curves.
- Estimating the yield curve.
- Sensitivity of present values.

Definition of yield

- If $P(t)$ is the market price of a “zero-coupon bond” that pays the risk-free amount of €1 at time t, its yield y is defined by the equation:
  $P(t) = e^{-yt}$
- The yield of the zero-coupon bond is defined as:
  $y(t) = -\frac{1}{t} ln\left(P(t)\right)$
- $y(t)$ is called the “spot rate” or “zero rate” for maturity t.

The chicken and the egg

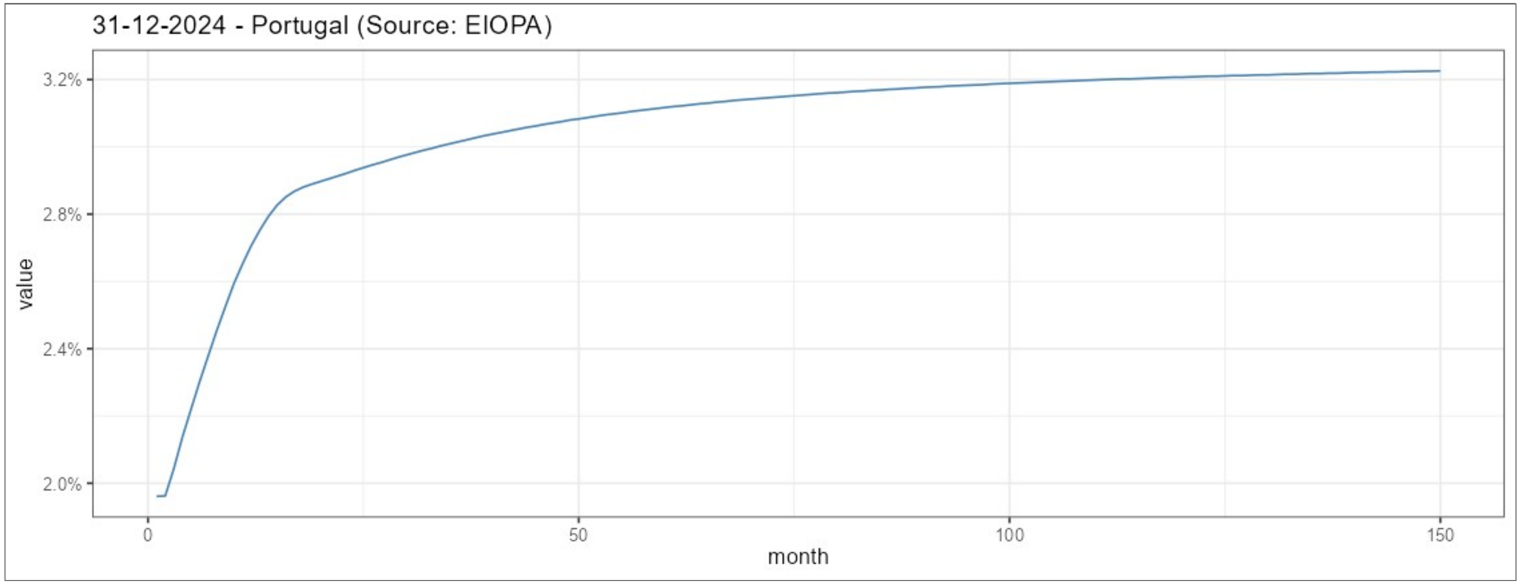
> **Note**
>
> The `yield` is just a way of expressing the `price`.

- $y(t)$ is also called the `spot rate` or zero rate for maturity t.
- the `current` yield curve describes the yields of `notional` zero-coupon bonds of €1, due at different times in the future.
- `Current` because the market price changes every day.
- `Notional` because there aren't zero coupon bonds for every maturity.

Yield curve examples

(see R script 1.Example yields.R)

Portugal



Discounting

- Assume that the yield curve $\{y(t) : t > 0\}$ is known.
- The arbitrage-free market value of a risk-free, future cashflow $\{c\left(t_1\right), c\left(t_2\right), \ldots, c\left(t_n\right)\}$ is:

$$B = \sum_{i=1}^{n} P\left(t_i\right) c\left(t_i\right) = \sum_{i=1}^{n} e^{-y(t_i)t_i} c\left(t_i\right)$$

- Every payment is valued separately as a zero-coupon bond.

Yields are strange

Consider this:

- The spot rate $y(t)$ at maturity t is the `constant` yield rate in the interval $(0, t)$ that reproduces the observed price $P(t)$ of €1 payable at time t.
- At the same time we are aware that the yield curve is `not constant`!

Forward rates

- The `forward rate` $y_F(t)$ is the implied yield in the infinitesimal time interval $(t, t+dt)$, defined consistently with the spot rate.
- The spot rate is the `average` of forward rates in the interval $(0, t)$.

Forward rates

- Forward rates $y_F(t)$ are defined by spot rates through the equation $\int_0^t y_F(s)ds = y(t) \cdot t.$
- Assuming differentiability, we have $y_F(t) = y(t) + t \cdot y'(t).$

Annual compounding

- Let n be an integer.
- Let $P(n)$ be the market price of a `zero coupon bond` that pays the risk free amount of €1 at time n.
- Then the yield i with annual compounding is defined by $P(n) = (1+i)^{-n}$.
- The yield of zero coupon bonds can be explicitly calculated: $i = i(n) = P(n)^{-\frac{1}{n}} - 1 = e^{y(n)} - 1$

Annual compounding

Note

Recall the relationship between yield with annual compounding (i) and yield with continuous compounding (y):

$$i = e^y - 1$$

$$y = ln(1 + i)$$

Why continuous compounding?

- Continuous compounding allows a `unified and simple notation`, e.g.
  $$P(t) = exp(-y(t) \cdot t) = exp\left(\int_0^t y_F(s)ds\right)$$
  regardless of wether t is an integer (whole year) or not.
- In this lecture we will use continuous compounding.
- In the financial press, annual and semi-annual compounding is common.

Bonds

- A bond can be defined in general as *"a promise to make a series of payments of specified size, at specified times in the future"*.
- Let us denote by $c(t_i)$ the payment due at time t_i, for $i = 1, \ldots, n$.
- We assume that bonds have no credit risk.

Bond yield

- Let $\{c(t_i) : i = 1, \ldots, n\}$ be the payments stipulated by a bond.
- Let B be the price being paid for the bond in the market.
- The average yield $\bar{y}$ of the bond is defined (implicitly) by

$$B = B(\bar{y}) \stackrel{!}{=} \sum_{i=1}^{n} e^{-\bar{y} t_i c(t_i)} \stackrel{def}{=} \int_0^{\infty} e^{-\bar{y} t} dC(t)$$

- The average bond yield is well-defined if all payments are non-negative.

Bond yield example

We are at the 31st December 2024. We will compute `forward rates` compatible with the (continuous) assumed market yield, the `price of the bond` and its `yield` (annual and continuous).

- Face value: 100
- Annual coupons: 5%
- Maturity: 5 years
- Market assumptions for Portugal by EIOPA

Coupon (%) 5

Face Value 100

time t	spot rate	price of €1	cash flows	PV c.flows
0.00	0.000%	1.00	0.00	0.00
1.00	2.707%	0.97	5.00	4.87
2.00	2.930%	0.94	5.00	4.72
3.00	3.015%	0.91	5.00	4.57
4.00	3.075%	0.88	5.00	4.42
5.00	3.117%	0.86	105.00	89.85

- Bond price: 108.4179946

Yield curve estimation

Estimating the market yield curve by `replication`

- Assume that you know the market prices $B_1, \ldots, B_n$ of n different government bonds.
- Define the payoff matrix

$$\mathbf{C} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} Payments\ of\ bond\ 1 \\ \vdots \\ Payments\ of\ bond\ n \end{pmatrix}$$

- Some of the c_{ij} may be `zero` but all bonds' total payments must be restricted to the time points $t_1, \ldots, t_n$.

Yield curve estimation - replication

- We construct a portfolio $(w_1, \ldots, w_n)$ that `replicates` the cash flow of a zero-coupon bond at maturity t_j:
  $(w_1, \ldots, w_n)\mathbf{C} \overset{!}{=} (0, \ldots, 0, 1, 0, \ldots, 0)$
- The equation is solved by
  $(w_1, \ldots, w_n) = (0, \ldots, 0, 1, 0, \ldots, 0)\mathbf{C}^{-1} = row_j\mathbf{C}^{-1}$
- Then, the price of the zero-coupon bond at maturity t_j is
  $P(t_j) = \sum_{i=1}^{n} w_i B_i$

Yield curve estimation - replication

- The `implied zero rate` $y(t_j)$ is given by solving $P(t_j) = e^{y(t_j)t_j}$
- In theory, finding yield curves is easy matrix algebra. In practice there are a number of problems. For example:
  - Not enough traded bonds to cover all time points.
  - Payments at other time points.
  - Lack of long term bonds.
- In practice you would use a software or the `risk-free rates delivered by EIOPA, Bloomberg` or others.

Example - Market assumption 31/12/2024

```
# A tibble: 15 × 5
    Bond `Mat. 31/12` `Face value` `Face val.` `Avg. yield`
   <dbl>        <dbl>        <dbl>       <dbl>        <dbl>
 1     1         2025          100        0.04       0.0219
 2     2         2026          100        0.04       0.0247
 3     3         2027          100        0.04       0.0255
 4     4         2028          100        0.05       0.0267
 5     5         2029          100        0.05       0.0281
 6     6         2030          100        0.05       0.0293
 7     7         2031          100        0.05       0.0305
 8     8         2032          100        0.05       0.0315
 9     9         2033          100        0.05       0.0324
10    10         2034          100        0.05       0.0329
11    11         2035          100        0.05       0.0332
12    12         2036          100        0.05       0.0335
13    13         2037          100        0.05       0.0338
14    14         2038          100        0.05       0.0341
15    15         2039          100        0.05       0.0343
```

Example - Payment Matrix

```
      2024 2025 2026 2027 2028 2029 2030 2031 2032 2033 2034 2035 2036 2037
 [1,]  104    0    0    0    0    0    0    0    0    0    0    0    0    0
 [2,]    4  104    0    0    0    0    0    0    0    0    0    0    0    0
 [3,]    4    4  104    0    0    0    0    0    0    0    0    0    0    0
 [4,]    5    5    5  105    0    0    0    0    0    0    0    0    0    0
 [5,]    5    5    5    5  105    0    0    0    0    0    0    0    0    0
 [6,]    5    5    5    5    5  105    0    0    0    0    0    0    0    0
 [7,]    5    5    5    5    5    5  105    0    0    0    0    0    0    0
 [8,]    5    5    5    5    5    5    5  105    0    0    0    0    0    0
 [9,]    5    5    5    5    5    5    5    5  105    0    0    0    0    0
[10,]    5    5    5    5    5    5    5    5    5  105    0    0    0    0
[11,]    5    5    5    5    5    5    5    5    5    5  105    0    0    0
[12,]    5    5    5    5    5    5    5    5    5    5    5  105    0    0
[13,]    5    5    5    5    5    5    5    5    5    5    5    5  105    0
[14,]    5    5    5    5    5    5    5    5    5    5    5    5    5  105
[15,]    5    5    5    5    5    5    5    5    5    5    5    5    5    5
      2038
 [1,]    0
 [2,]    0
 [3,]    0
 [4,]    0
 [5,]    0
```

Example - Clean market price B

```
# A tibble: 15 × 6
    Bond `Maturity 31.12. …` `Face value` Coupon `Average yield_annual`
   <dbl>               <dbl>        <dbl>  <dbl>                  <dbl>
 1     1                2025          100   0.04                 0.0219
 2     2                2026          100   0.04                 0.0247
 3     3                2027          100   0.04                 0.0255
 4     4                2028          100   0.05                 0.0267
 5     5                2029          100   0.05                 0.0281
 6     6                2030          100   0.05                 0.0293
 7     7                2031          100   0.05                 0.0305
 8     8                2032          100   0.05                 0.0315
 9     9                2033          100   0.05                 0.0324
10    10                2034          100   0.05                 0.0329
11    11                2035          100   0.05                 0.0332
12    12                2036          100   0.05                 0.0335
13    13                2037          100   0.05                 0.0338
14    14                2038          100   0.05                 0.0341
15    15                2039          100   0.05                 0.0343
# ℹ 1 more variable: `market price B` <dbl>
```

Example - market yield curve

	time	price of €1	spot rate
1	1	0.9785475	2.169%
2	2	0.9522118	2.448%
3	3	0.9269405	2.529%
4	4	0.8994891	2.648%
5	5	0.8693488	2.800%
6	6	0.8390086	2.926%
7	7	0.8074717	3.055%
8	8	0.7765943	3.160%
9	9	0.7453737	3.265%
10	10	0.7180624	3.312%
11	11	0.6922223	3.344%
12	12	0.6669583	3.375%
13	13	0.6423107	3.405%
14	14	0.6183099	3.434%
15	15	0.5949776	3.462%

Yield curve estimation - bootstrapping

- Assume you have bonds $i = 1, \ldots, n$.
- Bond nr. i matures at time t_i, pays coupon c_i and its current market price is B_i.
- All bonds have principal 1.

1. Solve for the first bond

$$B_1 = (1 + c_1)\, P(t_1) \Rightarrow P(t_1) = \frac{B_1}{1 + c_1} = e^{-y(t_1)t_1}$$

Yield curve estimation - bootstrapping

2. Solve for each subsequent bond

$$\begin{aligned} B_m &= \quad c_m \underbrace{\textstyle\sum_{i=1}^{m-1} P(t_i)}_{\textbf{known}} + (1 + c_m) \underbrace{P(t_m)}_{\textbf{unknown}} \\ &\Rightarrow \quad P(t_m) = \frac{B_m - c_m \sum_{i=1}^{m-1} P(t_i)}{1 + c_m} = e^{-y(t_m) t_m} \end{aligned}$$

Present value sensitivity

- Let's assume we have
- a future cash flow $\{C(t) : t > 0\}$;
- the current yield curve $\{y(t) : t > 0\}$.
- The present value of the cash flow is
$$B(y) = \int_0^\infty e^{-y(t)t} dC(t)$$
- How will the **PV** of B change if the yield curve changes?
- The easy answer: Calculate it!
- The traditional answer: Estimate it!

Duration and convexity 1

- The derivative of the PV with respect to a uniform shift in the entire yield curve is $B'(y) = \lim_{\Delta\bar{y}\to 0} \frac{1}{\Delta\bar{y}} \left(\int_0^\infty e^{-(y(t)+\Delta\bar{y})t} dC(t) - \int_0^\infty e^{-y(t)t} dC(t) \right)$
- The first and second derivative of the PV are $B'(y) = -\int_0^\infty t e^{-y(t)t} dC(t),\ B''(y) = \int_0^\infty t^2 e^{-y(t)t} dC(t)$

Duration and convexity 2

- Using the `Taylor expansion` we approximate the change in present value if the yield curve shifts:
  $$B(y+\Delta\bar{y}) - B(y) \approx B'(y)\Delta\bar{y} + \frac{1}{2}B''(y)(\Delta\bar{y})^2$$
- Define `duration` of the cash flow as
  $D = D(y) = -B'(y)/B(y)$
- Define `convexity` of the cash flow as
  $C = C(y) = B''(y)/B(y)$

Duration and convexity 3

- Rewrite the Taylor expansion in the following way:
$$\frac{B(y+\Delta\bar{y})-B(y)}{B(y)} \approx -D(y)\Delta\bar{y} + \frac{1}{2}C(y)(\Delta\bar{y})^2$$
- In words: One can approximate the `relative change` in the **PV** of the cash flow when the yield curve is shifted `uniformly` by a small amount.
  - To first order: minus the yield change $\Delta\bar{y}$, times duration.
  - To second order: Same as above, plus the squared yield change times one-half convexity.

Example PV Sensitivity 1

Consider a bond with face value of €100, maturity of 5 years and yearly coupons of 5%.

```
  duration convexity
1 4.567348  21.98331
```

- We will value it under market assumptions (€110.07) and estimate the effect of a parallel yield perturbation:
- increase of 1% - `1st. order` €105.04, `2nd order` €105.16.
- decrease of 1% - `1st. order` €115.09, `2nd order` €115.22

Properties of duration and convexity 1

- The duration and convexity of a `zero-coupon` bond payable at time t are t and t^2, `independent of the yield`.
- Duration and convexity decrease when the yield increases.
- For a given duration, convexity increases with the dispersion of the flow, because
$$\underbrace{\frac{1}{B(y)}\int_0^\infty (t-D(y))^2 e^{-y(t)t}\,dC(t)}_{\textbf{Dispersion, similar to variance}} = C(y) - D^2(y)$$

Properties of duration and convexity 2

- The duration/convexity approximation is an `easy way to estimate the sensitivity` of a cash flow's **PV** to small changes in the yield curve.
- The average duration/convexity of a portfolio is the **PV**-weighted average of the constituent durations/convexities. This makes those quantities easy to use.
- The duration/convexity approximation is `valid only` when there is a `parallel` shift in the yield curve.

Properties of duration and convexity 3

- The duration/convexity approximation `does not` tell us what change in the present value to expect, should different parts of the yield curve change by different amounts or even in different directions.

Different concepts of duration

- `Macaulay Duration`: The time weighted PV divided by the PV.
- `Modified Duration`: Macaulay Duration divided by $1 + i(n)/n$, where n is the compounding frequency.
- `Effective Duration`: Calculated by shocking the yield curve up and down by some change in PV.
- `Dollar Duration`: $DD(y) = -B'(y) = B(y)D(y)$
- `Dollar Convexity`: $DC(y) = B''(y) = B(y)C(y)$

Speaker notes